Computer-aided Explorations of Modular Spaces of Real Polynomials: Giving Geometric Life to Routine Algebra

Gabriel Katz
e-mail: gabkatz@gmail.com
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Vladimir Nodelman
e-mail: nodelman@hit.ac.il
Holon Institute of Technology
Holon, 52 Golomb St., 58102, Israel

Abstract

We suggest a novel visual approach to teaching topics in elementary algebra from a viewpoint that is characteristic of modern mathematics. It puts emphasis on families of mathematical like-objects, rather than on a prototypical object from a given family. Thanks to rich computer-supported virtual mathematical environments (like VisuMatica*), it becomes feasible to convey this advanced circle of ideas to students with a modest mathematical background. In the paper, we us illustrate our approach with a few basic examples; they revolve around the problem of solving quadratic and cubic equations.*

1. The Root to Coefficient Map for Quadratic Polynomials

Consider the problems of multiplying linear polynomials, factoring quadratic polynomials, solving quadratic equations, making sense of the quadratic formula. Think of endless exercises involving forms like $(x - 2)(x + 7)$, exercises that promote rules like "FOIL", exercises in determining the zeros of polynomials, exercises in solving quadratic equations like $x^2 - 2x - 3 = 0$ by factoring, by completing the square, by using the quadratic formula. Students meet them first in a non-visual space of symbolic expressions. Later, when representations do get visual, they struggle to make appropriate connections between a space of one kind (symbolic forms) and a space of another (*xy*-graphic representations). Think of the equivalent forms that we ask them to comprehend: $(x - 3)^2 - 1$, $x^2 - 6x + 8$, $(x - 2)(x - 4)$. Think of the connections that we ask them to make between changes in symbolic form and changes in *xy*-graphs.

We propose to look at this material from a radically different point of view. Consider the monic polynomial $f(x) = (x - 3)(x - 2)$. It can also be written in the form $f(x) = x^2- 5x + 6$. The coordinate pair (3, 2) represents the polynomial in one way; the coordinate pair (-5, 6) the same polynomial in another. Thus, points in the coordinate plane can either represent the *zero* (or *root*) *form* or the *coefficient form* of the quadratic.[1] The fact that we can always convert the root form into a coefficient form means that we can define a *map* from the root space to the coefficient space.

VisuMatica, a comprehensive software for visualizing mathematics, is an appropriate tool for the task. It is a brainchild of the second author. All the figures in this paper are produced with the *VisuMatica* help.

Given enough time, students have little trouble in defining the root-to-coefficient map V: $(r_1 \cdot r_2) \to (-r_1 - r_2, r_1 r_2)$. There is a good chance that asking them to construct this map is a more appealing way to get them to connect the roots to the coefficients. Once the map V is defined, the

[1] We are dealing only with monic polynomials, that is, with those for which the coefficient of x^2 is 1.

student can investigate its geometry. The color-coding in *VisuMatica* helps to identify points in the domain with their images in the range: for 1-to-1 maps F, the color of each point z and its image F(z) are identical; for other maps, the color correspondence is more subtle and layered. Placing the cursor in the range window allows the user to see its preimage(s) in the domain window. In the coefficient (range) window, the screen snap in Figure 1 shows the result of mapping points from the root (domain) window. The student has created the diagonal line $\{r_1 = r_2\}$ in the domain: its V-image seems to be the boundary between the points that are hit by the map V and those that are not.

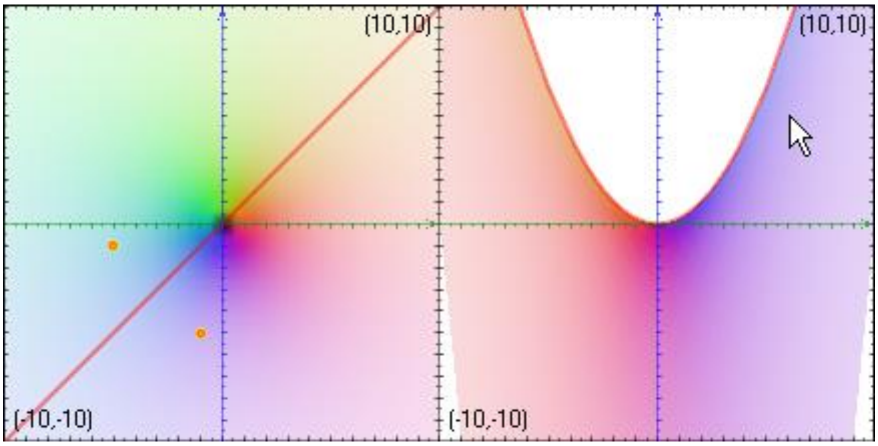

Figure 1: *the root-to-coefficient map V*

There are lots of issues to explore. What quadratic polynomials are represented by the points that are not in the image of V? What is the bc-equation of the (parabolic?) boundary D of the shaded region? (We call this D the *discriminant curve*.) What is the image of all quadratic polynomials that have multiple zeros? Why do all the images of domain points fall on or below the discriminant curve D? How is this related to the quadratic formula? As one experiments with the V-images, it becomes obvious that different points in the root space (the points (2, 3) and (3, 2) for example) map to the same point in the coefficient space. Is there a subset of the domain for which the map is 1-to-1? Finally, is there an *inverse* map that sends the appropriate points (b, c) to the real points (r_1, r_2)?

Notice that all of the questions relate to two representational spaces that are visual and similar. Contrast this with standard questions connecting representations that are either both symbolic, or an odd mating of a symbolic to a coordinate xy-representation. In a second year algebra course, we may suppose that the student has already made a connection between linear equations and their graphs and probably quadratic equations and their graphs.

Once a student determines that the discriminant curve is the image of the line $\{r_2 = r_1\}$, then the parametric equations $b = -2r_1$ and $c = r_1^2$ generate the discriminant equation $c = b^2/4$. It is important to observe that only the bc-points that lie on or beneath this parabola represent equations that have *real* roots.

If a student uses *VisuMatica*'s zoom-out feature to take an expanded view of the coefficient space, the region above the parabola seems to shrink.

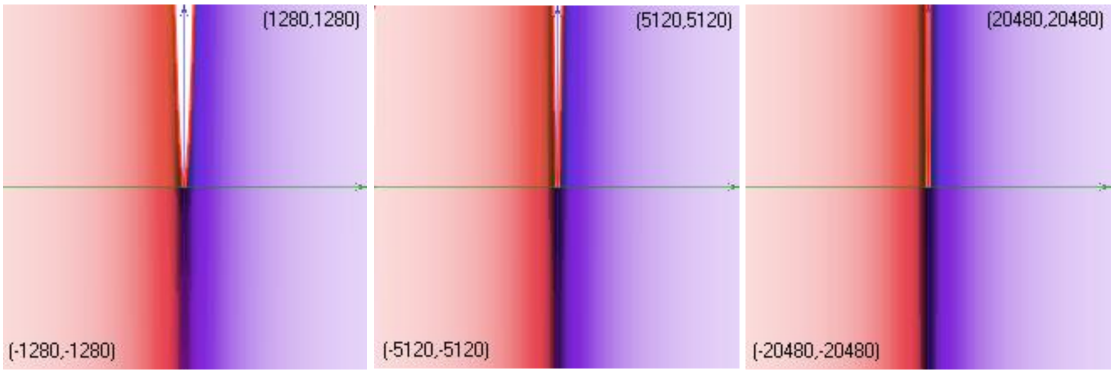

Figure 2: *Solving real quadratic equations probabilistically*

As one zooms out in stages, the space above the discriminant curve occupies proportionately less of the square window. This says something important about the solutions of quadratic equations having real coefficients. Here are a couple of questions that can get students thinking about this.

> Estimate the probability that the quadratic equations represented by points in a 10,000 by 10,000 region of the coefficient space have real zeros?
>
> What happens to this probability as the region of the *bc*-plane (shown in the window) increases in size?

It is surprising to discover that, observed at a large enough scale, almost all quadratic equations with real coefficients are solvable over the real numbers!

The geometry of the plane is useful in classifying and organizing the modular space it represents. In the case of the root space, we have investigated the image of the line $\{r_2 = r_1\}$, which represents the set of quadratic polynomials having one root of multiplicity 2. There are two interesting families of quadratic polynomials: the vertical lines represent the family of quadratic polynomials having a known "first" root; the horizontal lines represent the family of quadratic polynomials having a known "second" root. The character of the images of these lines under the root-to-coefficient map V ought to be of interest.

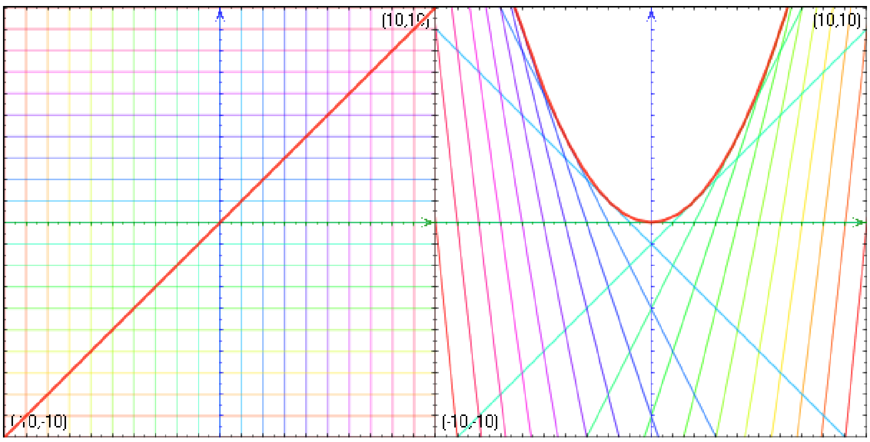

Figure 3: *The enveloping geometry of the real root-to-coefficient map V*

VisuMatica allows its users create a grid in the domain window. The screen snap in Figure 3 shows the image of a 20 by 20 grid under the root-to-coefficient map V. The result is quite startling. Why do the images of vertical and horizontal lines in the domain space all appear to be lines *tangent* to the discriminant curve D? In other words, is D the *enveloping curve* for this family of straight lines?

If $r_1 = 3$ is a root, then substituting 3 in the polynomial equation gives $3^2 + b(3) + c = 0$. The resulting linear equation $c = -3b - 9$ in the variables b and c describes the image of the line $r_1 = 3$. Try calculating the V-image of a few more vertical lines. The slope of the line (-3 in the case $r_1 = 3$) seems always to be the opposite of the root; the y-intercept of the line (-9 in the case $r_1 = 3$) seems always to be the opposite of the square of the root. Is it possible to prove this?

This is a lovely opportunity to show how a change in viewpoint can pay off. Switch the roles of variables and parameters. A vertical line in the root space is one for which the first root is some number k and therefore it has an equation $\{r_1 = \mathrm{k}\}$. However, the fact that k is a root means that $\mathrm{k}^2 + b\mathrm{k} + c = 0$. But this is the equation of a line in the bc-plane: the fixed root k is now just a coefficient! The line $c = -\mathrm{k}b - \mathrm{k}^2$, expressing c as a function of b, represents the image of $\{r_1 = \mathrm{k}\}$ under the root-to-coefficient map. Indeed, its slope is -k!

One way of arguing that the image lines are *tangent* to the discriminant curve depends on showing that the image of $\{r_1 = \mathrm{k}\}$ must contain a point on the discriminant parabola and does not contain points above it. We think that many second year algebra students are capable of making this argument.

By the way, here is a nice problem for an enterprising student: devise a graphic method for solving quadratic equations using tangents to the discriminant curve $D = \{c = b^2/4\}$ in the bc-plane. The solution makes use of the fact that any two tangent lines to D that pass through a coefficient point (b, c) must have slopes that are the negative of the two roots of the equation $x^2 + bx + c = 0$. We envision a neat project: to design and built an analogue quadratic-equation-solving machine.

2. The Inverse Map

Many students know the quadratic formula, but nobody likes it… We aim to change their view of this mathematical contraption by treating the formula as a *coefficients-to-roots map*! Let us order real roots by their magnitude: $r_2 \le r_1$. This has the effect of restricting the root space to points on or under the line $\{r_2 = r_1\}$. In other words, the range of the coefficient-to-root map is restricted to the subspace $\{r_2 \le r_1\}$. (This provides a nice opportunity to deal with some of the issues relating to maps and their inverses.) Under the restriction $b^2 - 4c \ge 0$ in the coefficient bc-space, the quadratic formulas

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad\qquad r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

generate a coefficient-to-root map W.

Once students define the W map, *VisuMatica* offers a rich opportunity to explore the connection between real coefficients and real roots. The image of the line $\{r_2 = r_1\}$ under the root-to-coefficient map V is the discriminant parabola D. Therefore the image of D under the coefficient-to-root map W should be the line $\{r_2 = r_1\}$.

An image of a square in the domain reveals an odd condensation of roots in the range space (Figures 5 and 6).

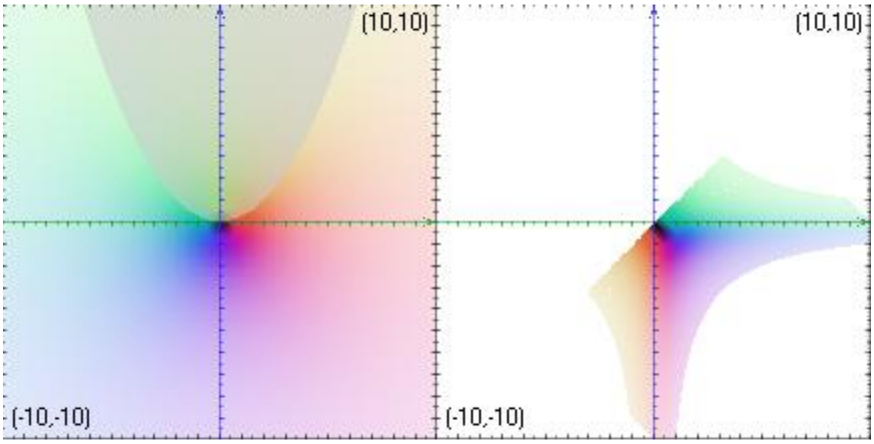

Figure 4: *The real coefficient-to-root map W*

The pattern in the range window is quite surprising: there seems to be empty space below the line $\{r_1 = r_2\}$. Furthermore, points in the root space occur in a relatively narrow belt bounded by curves that look like hyperbolas. What is going on? Increasing the scale seems to make things worse: it looks as if most points in the root space have no preimages in the coefficient space—an obvious absurdity. Furthermore, seen at this scale, one of every pair of roots in the image seems always to be approximately 0!

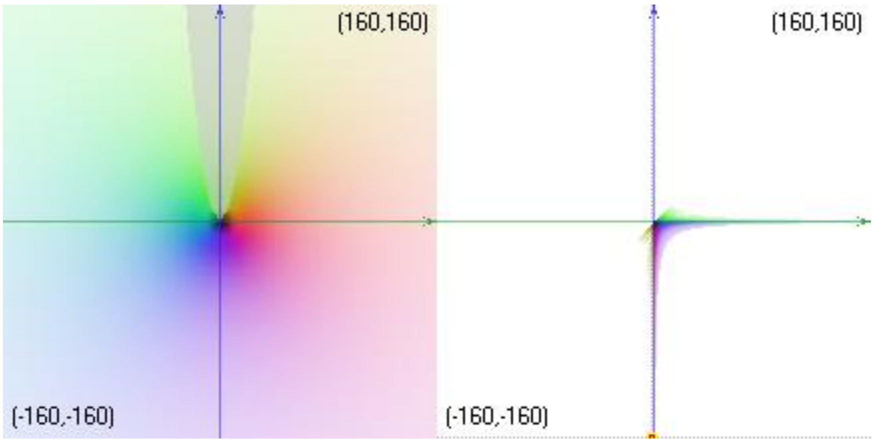

Figure 5: *The same map W viewed at a different scale*

However, increasing the scale of the coefficient-space window serves to demonstrate that there are indeed points very far out in the coefficient space that map to points in the seemingly empty regions in the root-space window. The fact that $c = r_1 r_2$ also indicates why the roots concentrate the way they do. The roots in the range window must lie between four branches of two hyperbolas. When the range scale is large, this region appears to be very close to the axes.

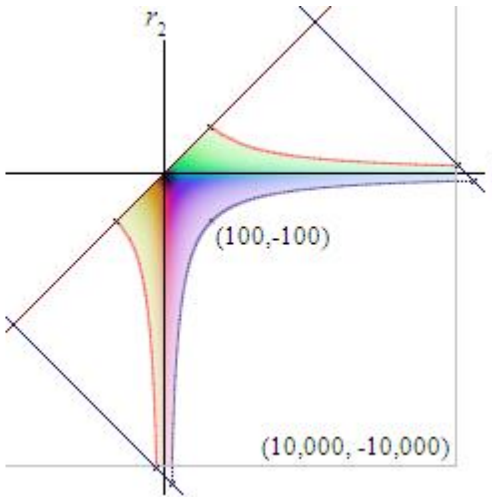

Figure 6: *Resolving the scale paradox for the map W*

3. Modular Spaces of Cubic Polynomials

Next we launch a similar graphic investigation of the universe of cubic (monic) polynomials. It is a 3-dimensional space, so *VisuMatica* is still able to handle the visualization challenge. However, the geometry becomes more intricate and the relevant mathematic a bit more advanced and interesting.

If r_1, r_2, and r_3 are the roots of a cubic equation $x^3 + bx^2 + cx + d = 0$, then $x^3 + bx^2 + cx + d = (x - r_1)(x - r_2)(x - r_3)$. As a result, we are getting the formulas:

$$b = -(r_1 + r_2 + r_3), \quad c = r_1r_2 + r_2r_3 + r_3r_1, \quad d = -r_1r_2r_3.$$

They define a root-to-coefficient map V: $(r_1, r_2, r_3) \longrightarrow (b, c, d)$ whose geometry reflects various aspects of the problem of solving cubic equations. First, let us assume that all the coefficients and roots are real. Then V maps $\mathbf{R}^3_{\text{root}}$, the space of roots, to $\mathbf{R}^3_{\text{coef}}$, the space of coefficients.

Before examining the geometry of the map V: $\mathbf{R}^3_{\text{root}} \longrightarrow \mathbf{R}^3_{\text{coef}}$, with the help of *VisuMatica*, we can visualize its *slice* over the plane $\{b = \text{const}\}$. An interesting slice $\{b = 0\}$ produces the space of *depressed* cubic polynomials. The roots of a depressed polynomial satisfy the relation $r_1 + r_2 + r_3 = 0$. Therefore, r_1 and r_2 can be chosen as independent variables, and $r_3 = - r_1 - r_2$. Replacing r_3 with $- r_1 - r_2$ in the formulas for V, we get a new map V_{depr}: $\mathbf{R}^2_{\text{root}} \longrightarrow \mathbf{R}^2_{\text{coef}}$ from the r_1r_2-plane to the cd-plane. This map is easier to visualize than its 3D analogue V. Figure 7 depicts the image of V_{depr}. Note that D, the image of the diagonal line $\{r_1 = r_2\}$ under the map V_{depr}, is a cubic curve whose parametric equation is given by the formulas $\{c = - r_1^2 - r_2^2 - r_1r_2, \ d = - r_1^2r_2 - r_2^2r_1\}$. Equivalently, the curve D is given by the equation $\{4c^3 + 27d^2 = 0\}$. It is called the *discriminant curve*.

Experimenting with the *VisuMatica*, we see that the curve D is exactly the boundary of the domain $V_{\text{depr}}(\mathbf{R}^2_{\text{root}})$. Moreover, $V_{\text{depr}}(\mathbf{R}^2_{\text{root}})$ is given by the inequality $\{4c^3 + 27d^2 \leq 0\}$. As in the case with quadratic equations, we encourage students to interpret these phenomena. For instance, what is the significance of the empty region $\{4c^3 + 27d^2 > 0\}$? Does it mean that the equations $x^3 + cx + d = 0$ have no real solutions when $4c^3 + 27d^2 > 0$? Of course, this would contradict to the basic theorem of real algebra, which claims that any real polynomial of *odd* degree has a real root! After a bit of contemplation, we realize that the region $\{4c^3 + 27d^2 > 0\}$ represents polynomials with a *single* simple real root (the other two roots form a complex-conjugate pair). Thus, the region $\{4c^3 + 27d^2 \leq 0\}$ represents polynomials with all their roots being real.

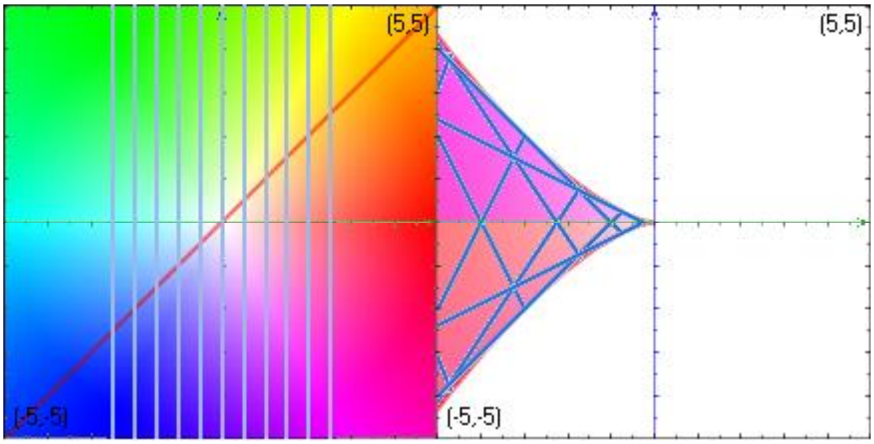

Figure 7: *The image of the depressed real root-to-coefficient map* V_{depr}. *Note the pattern formed by the* V_{depr}*-images of the vertical lines*

Let us glance at the V_{depr}-images of vertical lines $\{r_1 = \text{const}\}$ in the r_1r_2-plane. They form a remarkable pattern of rays in the cd-plane: again, the rays seem to be *tangent* to the discriminant curve D! Even the fact that V_{depr} maps a vertical line to a ray merits an explanation; after all, V_{depr} is a cubic polynomial map. Thus we should expect that a generic line should be mapped to a cubic curve. For example, the points in the V_{depr}-image of the line $\{r_1 = 5\}$, evidently must satisfy the equation $\{5^3 + 5c + d = 0\}$. The latter is an equation of a line residing in the cd-plane. Note that it slope is -5, minus the root $r_1 = 5$, a phenomenon already familiar from the investigations of quadratic map V. It takes a bit of calculus to verify that such a line is tangent to the curve D at a point where they intersect. This observation suggests a construction of another analogue equation-solving device: the algebraic problem of solving the cubic equation $x^3 + cx + d = 0$ is equivalent to the geometric problem of finding lines that pass through the point (c, d) and are tangent to the discriminant curve $\{4c^3 + 27d^2 = 0\}$! When $\{4c^3 + 27d^2 < 0\}$, there are three such tangent lines; when $\{4c^3 + 27d^2 = 0\}$ and $(c, d) \neq (0, 0)$, there are two tangent lines; when $\{4c^3 + 27d^2 > 0\}$, the tangent line is unique.

What about solving depressed cubic equations $\{x^3 + cx + d = 0\}$ probabilistically? Specifically,

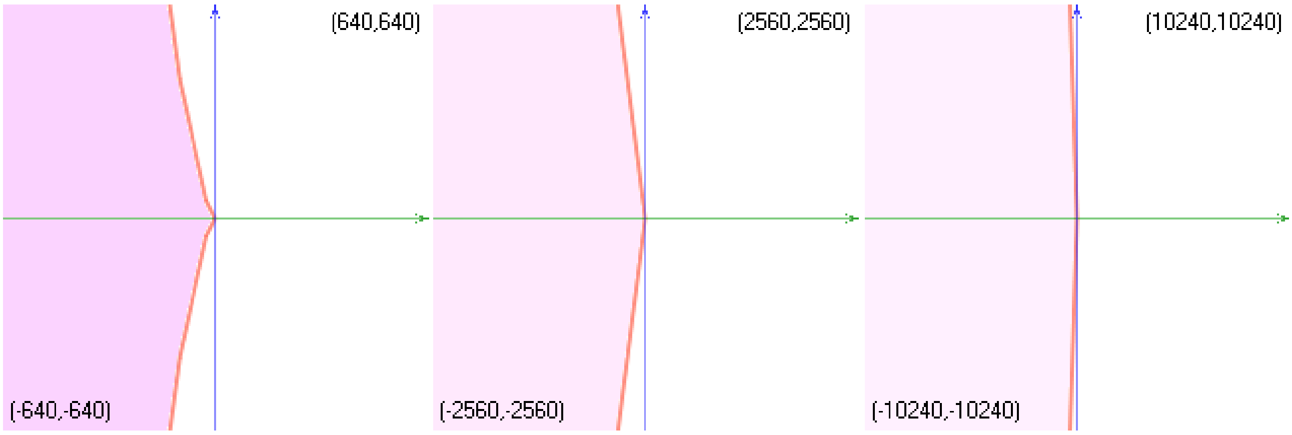

Figure 8: *Solving real cubic equations* $x^3 + cx + d = 0$ *probabilistically*

what is the probability that a randomly chosen equation $x^3 + cx + d = 0$ has three real roots, and what is the probability that it has a single real root? Figure 8 provides the answer: the probability is 1/2 and 1/2, respectively!

As in the case of quadratic equations, there is a cumbersome formula that solves cubic equations in radicals. It is commonly called the Cardano Formula. Here is a brief reminder how the formula works. Let $\alpha \neq 0$ be any complex number from the set $\{-c/2 + (d^2/4 + c^3/27)^{\{1/2\}}\}^{\{1/3\}}$ (in general, the cardinality of this set is 6). Let $\beta = -p/(3\alpha)$. Then the complex solutions of the equation $z^3 + cz + d = 0$ are: $\alpha + \beta$, $\omega\alpha + \omega^2\beta$, $\omega^2\alpha + \omega\beta$□ where $\omega = -1/2 + i\sqrt{3}/2$. Paradoxically, when the equation has *three* real roots (when $4c^3 + 27d^2 < 0$), we are forced to pick complex α and β; however, when only *one* real solution is available, α and β are real numbers, and so is the solution $x = \alpha + \beta$!

The Cardano algorithm implies that $c = -3\alpha\beta$, and $d = -\alpha^3 - \beta^3$. These relations motivate us to introduce the real *Cardano map* C: $\mathbf{R}^2_{\alpha\beta} \longrightarrow \mathbf{R}^2_{cd}$, defined by the formula $c = -3\alpha\beta$, $d = -\alpha^3 - \beta^3$. Its image is shown in Figure 9. Note that $C(\mathbf{R}^2_{\alpha\beta})$ is *complementary* to the image of the real cubic root-to-coefficient map V (cf. Figure 7).

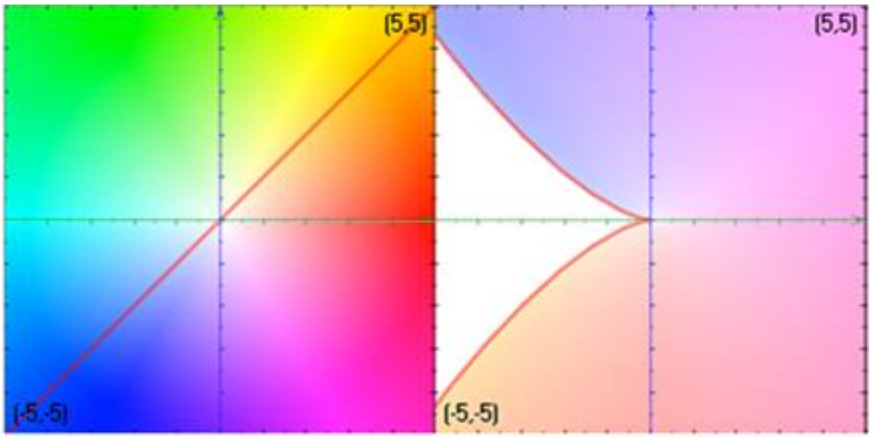

Figure 9: *The image of the Cardano Map* C: $\mathbf{R}^2_{\alpha\beta} \longrightarrow \mathbf{R}^2_{cd}$. *The diagonal line* $\{\alpha = \beta\}$ *is mapped to the discriminant curve* D.

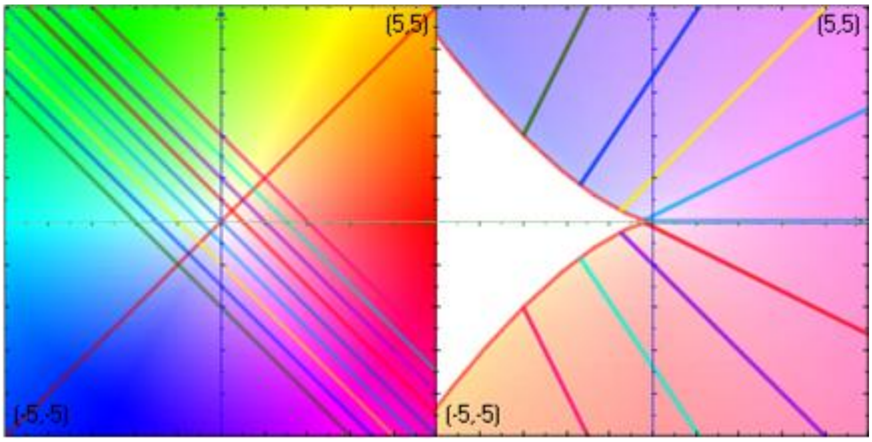

Figure 10: *Under the map* C: $\mathbf{R}^2_{\alpha\beta} \longrightarrow \mathbf{R}^2_{cd}$ *the lines* $\{x := \alpha + \beta = \text{const}\}$ *are mapped to the lines tangent to the discriminant curve* (*this is not evident from the figure: to verify this property, one needs to extend the rays*).

Let us take still another look at the geometry of depressed cubic equations. Consider the real surface $S = \{x^3 + cx + d = 0\}$ in the space $\mathbf{R}^3_{cdx}$. Figure 11 shows the shape of S.[2]

For each point (c^*, d^*) in the cd-plane, the intersection of the vertical line $\{c = c^*, d = d^*\}$ with the surface S consists of points (c^*, d^*, x^*), where x^* is a solution of the equation $x^3 + c^*x + d^* = 0$. So,

[2] in the figure, $p = c$, $q = d$, and $z = x$.

for (c^*, d^*) such that $4(c^*)^3 + 27(d^*)^2 < 0$, we should expect *three* intersections, for (c^*, d^*) such that $4(c^*)^3 + 27(d^*)^2 > 0$ — *one* intersection, and, for exceptional (c^*, d^*) such that $4(c^*)^3 + 27(d^*)^2 = 0$, — *two* intersections.

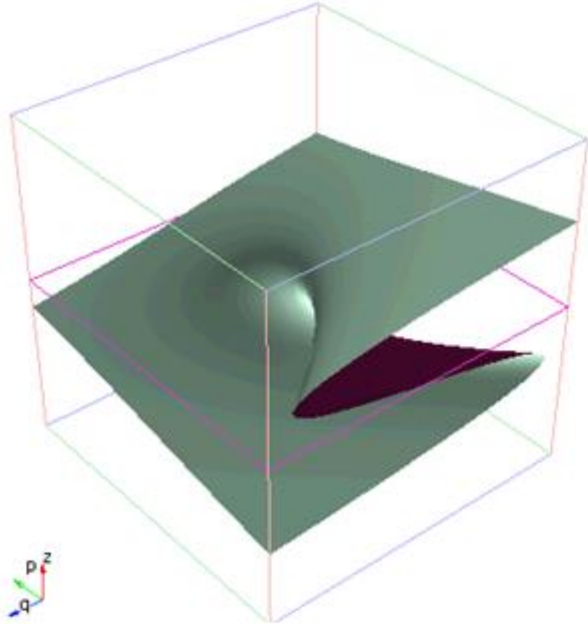

Figure 11: *The cubic surface* $S = \{x^3 + cx + d = 0\}$

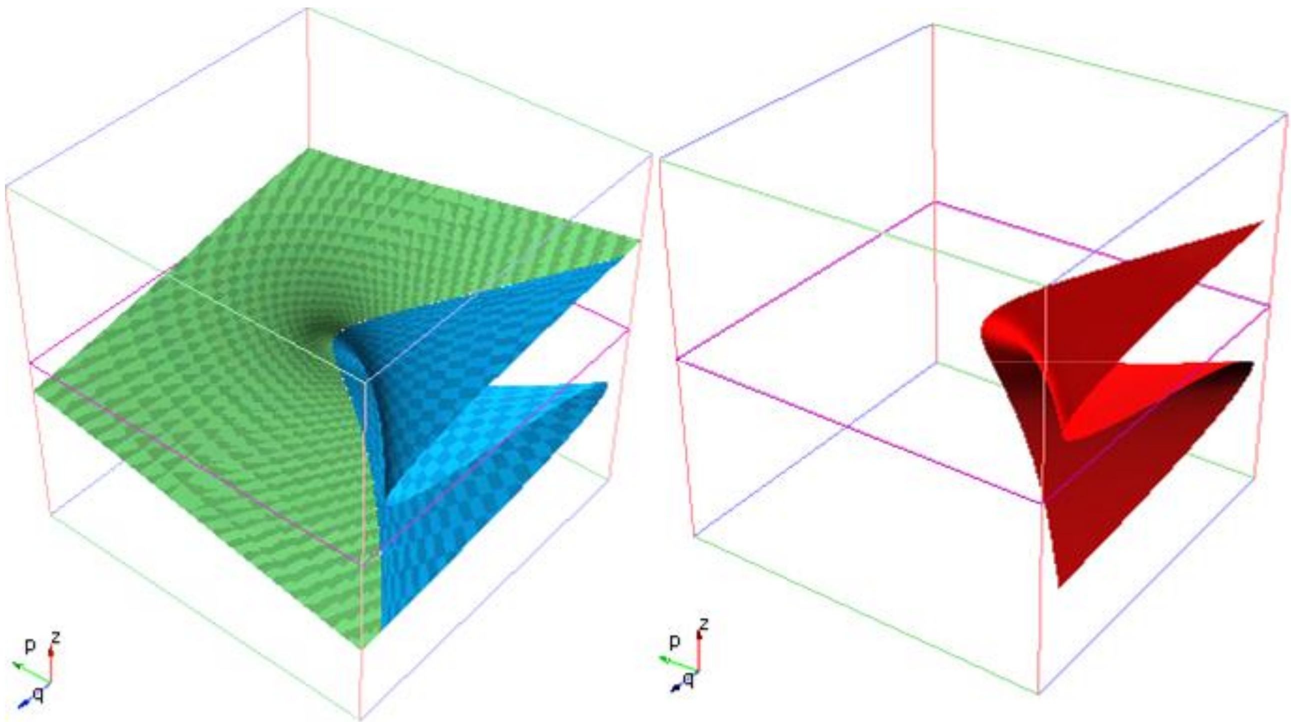

Figure 12: *The cubic surface* $S = \{x^3 + cx + d = 0\}$ *being assembled from two surfaces: the red surface is suspended over the domain* $\{4c^3 + 27d^2 < 0\}$, *the green surface—over the domain* $\{4c^3 + 27d^2 > 0\}$.

This surface S can be assembled from two complementary pieces as shown in Figure 12, the left picture. The first piece is suspended over the domain $\{4c^3 + 27d^2 < 0\}$ and is shown on the right. The second (green) piece is suspended over the domain $\{4c^3 + 27d^2 > 0\}$. Both pieces are sewed along a curve E that is suspended over the discriminant curve D in the cd-plane. Therefore, looking

at semitransparent S from far away in the direction of the x-axis, we will see the familiar images of Figures 7 and 9.
Finally, let us discuss how we can visualize the map V: $\mathbf{R}^3_{\text{root}} \longrightarrow \mathbf{R}^3_{\text{coef}}$ "in its full splendour".

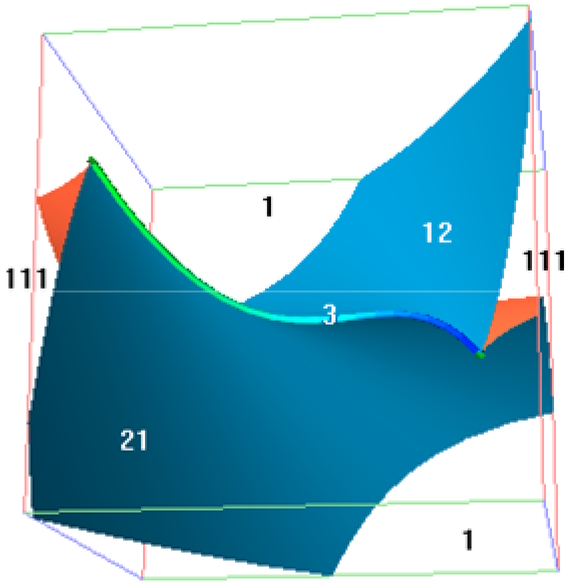

Figure 13: *The discriminant surface D^2 separates the coefficient space $\mathbf{R}^3_{bcd}$ into two chambers. The first one, marked with "1", represents polynomials with a single real root, the second one, marked with "111", represents polynomials with three real roots.*

Figure 13 depicts an important and revealing stratification in the space of real monic cubic polynomials. The strata are marked with the labels "1", "111", "12", "21", and "3". They represent the combinatorial patterns of real roots, taken with their multiplicities, and presented in the order in which they reside in the number line $\mathbf{R}$. For example, "111" stands for polynomials with three distinct real roots, $r_1 < r_2 < r_3$, "12" for the polynomials with two real roots, $r_1 < r_2$, where r_1 is of multiplicity 1 and r_2 is of multiplicity 2. In contrast, "21" stand for the polynomials with two real roots, $r_1 < r_2$, where r_1 is of multiplicity 2 and r_2 is of multiplicity 1. The label "3" represents polynomial with a single root of multiplicity 3. Those polynomials form a cubic curve C in $\mathbf{R}^3_{\text{coef}}$. The image of V is the chamber that is labelled by "111". It is bounded by the two-winged blue surface D^2. The wings are labelled by "12" and "21" and are attached to the cusp C. In fact, D^2 is generated by the lines that are *tangent* to the curve C!

VisuMatica can even help to get an insight into the modular spaces of the monic real polynomials of degree 4. Of course, it is difficult to visualize objects in the 4-dimensional space. So we restrict our attention to the 3-dimensional slice of the 4-dimensional reality, namely, to the space of depressed polynomials of degree 4, that is, to the real polynomials of the form $x^4 + cx^2 + dx + e$. The result is shown in Figure 14. As in Figure 13, the labels reflect the combinatorial patterns of real roots. The blue-orange discriminant surface Σ, called the *Swallow Tail*, represents polynomials with at least one multiple real root. It divides the modular space of polynomials into three chambers (marked with the three spheres). They correspond to polynomials with four real roots (labelled "1111"), two real roots (labelled "11"), and no real roots at all (labelled "∅"). Note that Σ (as the surface in Figure 13) is a *ruled* surface: as the left picture in Figure 14 testifies, it formed by a family of lines.

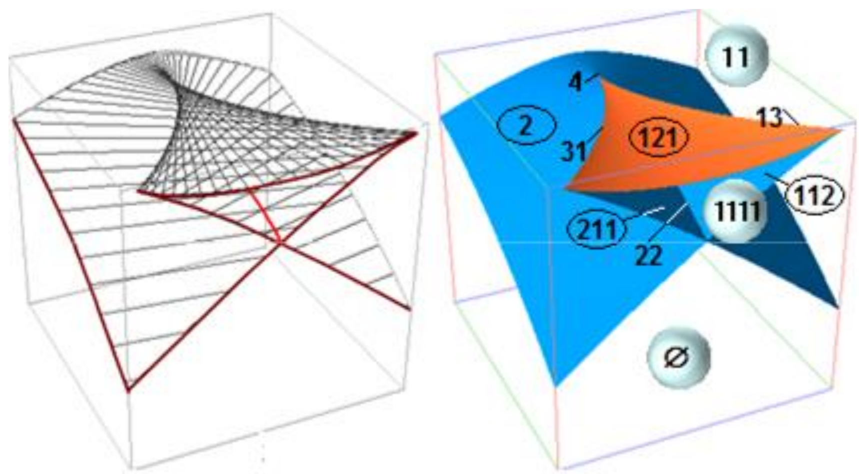

Figure 14: *The natural stratification in the space of depressed real polynomials of degree four*

4. Conclusions

Take the suggested investigations as an indication of the richness of the root-coefficient mapping environment. Investigating the maps can be approached on two levels—observational and algebraic. We think that the mapping environment with its strong visual component is likely to motivate and engage students and will lead to a better understanding of the most important issues relating to the solution of polynomial equations. The investigations suggested by Section 3 could also serve well and motivate the students who study the Multivariable Calculus. More importantly, these activities are designed to promote in students the view of mathematics adopted by its modern practitioners.

Further developments of these ideas could be found in the papers [1], [2], or in our forthcoming book "*The Shape of Algebra*" [3].

5. References

1. Feurzeig, W., Katz, G., Lewis, P., Steinbok, V., *Two-parameter Universes: Part I. Consider a Rectangular Point…,* International Journal of Computers for Mathematical Learning **5**: 169-178, 2000.
2. Feurzeig, W., Katz, G., Lewis, P., Steinbok, V., *Two-parameter Universes: Part II. Picture a Quadratic Polynomial…,* International Journal of Computers for Mathematical Learning **5**: 263-274, 2000.
3. Katz, G., Nodelman, V., *The Shape of Algebra: a Visual Exploration of Elementary Algebra and Beyond,* World Scientific, 2011.